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INVARIANT MEASURE IN THE PROBLEM OF A SYMMETRICAL BALL ROLLING ON A SURFACE†

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The problem of a symmetrical ball, rolling without slip on a fixed surface, is considered, along with some generalizations. It is proved that, under fairly general assumptions as to the nature of the external forces, the system admits of a smooth invariant measure. © 2003 Elsevier Ltd. All rights reserved.

The rolling of a ball without slip on a fixed surface is a classical example of a non-holonomic system [1]. As a rule, non-holonomic systems do not admit of an invariant measure, unlike Hamiltonian systems, which always have a standard invariant measure (moreover, a symplectic 2-form is preserved) [2]. One of the best known examples is perhaps the "Celtic stone" [3], whose rotations in one direction are asymptotically stable but in another direction are unstable (see, e.g. [4, 5]).

In the problem of a symmetrical ball rolling on a surface, investigations have been devoted in the main to special cases (special types of surface) in which the system has additional first integrals [6–8]; it has turned out that, in all the special cases considered, the system also has a smooth invariant measure. It will be proved below that the measure is preserved in the general case also, provided the external force is applied at the centre of the ball and satisfies certain additional conditions, which, in fact, mean that there is no dissipation in the system. A typical example is the force of gravity. This result is also valid in the "limiting case" in which the radius of the ball tends to zero.

The existence of an invariant measure is of fundamental significance for the dynamics of the system. If the external forces are potential forces, the system has a first integral – the energy integral. If the energy level is a compact manifold, one can confine the system to it (and it will preserve a certain measure on that manifold) and use ergodic theorems, such as Poincaré's recurrence theorem, Birkhoff's theorem, etc. [9]. Another obvious corollary of the existence of an invariant measure is the non-existence of asymptotic stability and attractors.

1. FORMULATION OF THE PROBLEM

Consider an absolutely symmetrical ball of radius r and mass m, rolling on a surface Σ defined (locally) by smooth functions in Euclidean three-space. By an absolutely symmetrical ball we mean a ball whose centre of mass is its geometrical centre and whose central tensor of inertia is spherical. An external force is applied to the ball, the point of application being the centre of the ball.

The equations of motion of the ball along the surface Σ are conveniently written, without using the D'Alembert–Lagrange equations, in the following form

$$m\dot{v} = R + F, \quad J\dot{\omega} = -rN \times R, \quad v = r\omega \times N \tag{1.1}$$

where J is the moment of inertia of the ball about its centre, v is the velocity of the centre, ω is the angular velocity of the ball, R is the reaction of the surface Σ, F is the external force, and N is the unit normal to the surface Σ at the ball's point of contact with the surface, pointing toward the centre of the ball. The non-holonomic constraint defined by the last equation of (1.1) is the condition that the ball is rolling without slip. We shall assume that the curvature of the surface Σ is sufficiently small (or that the radius r of the ball is small), so that the ball always touches the surface at one point only.

Remark 1. Generally speaking, in the context of a formal-anxiomatic definite of the motion of systems with constraints (such as, e.g., postulating the D'Alembert–Lagrange principle), both the physical meaning of the basic

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principles and the limits of applicability of the theoretical models remain unclear. It turns out that the classical non-holonomic model is a limiting case of the realization of a constraint by forces of viscous tension (see, e.g., [9–11]). For dry friction models in problems involving the rolling of rigid bodies see [12, 13].

Let Σ' denote the surface along which the centre of the ball is moving. Obviously, the original surface Σ is a wave front for Σ' (it is an envelope of the wave fronts [a sphere of radius r] for each point of Σ').

Proposition 1. The normal to the surface Σ' drawn from the centre of the ball coincides with N – the normal to Σ at the ball's point of contact with the surface.

This is a well-known fact in the theory of motion of wave fronts, but we shall present a more intuitive proof. Let the ball touch the surface Σ at a point x. Take the normal N to Σ at this point. The set of admissible velocity vectors of the centre of the ball is a plane orthogonal to this normal. But any velocity vector of the centre of the ball lies in the tangent plane to the surface Σ' at the point x' where the surface is cut by a straight line drawn parallel to N through the point x where the ball touches the surface Σ. Thus, the segment $[x, x']$ parallel to N will also be orthogonal to Σ'

Let us fix the surface Σ' and, varying the radius of the ball r, consider the surface Σ as variable. This is an equivalent (and generally more convenient) formulation of the initial non-holonomic problem, going back as far as Routh. The motion of the ball will be described, as before, by Eqs (1.1), and the unit vector N will be the normal to Σ' drawn from the centre of the ball (by Proposition 1); the centre of the ball is moving along that surface.

Put $w = r\omega$ and divide the second and third equations of system (1.1) by r. We obtain the equations

$$m\dot{v} = F + R, \quad M\dot{w} = -N \times R, \quad v = w \times N \tag{1.2}$$

which do not contain r – they are exactly the same as system (1.1) in which $r = 1$. Here $J = Mr^2$, where the number M is independent of r. In what follows we shall use system (1.2); the initial system (1.1) may be retrieved from (1.2) by the substitutions $w = r\omega$ and $J = Mr^2$.

Proposition 2. The equations of the limiting problem as $r \to 0$ are equivalent to system (1.2).

Indeed, Eqs (1.2) are invariant to variation of the radius r of the ball, since we have assumed that it is the surface Σ' that is held fixed. Thus, the limiting equations are equivalent to the initial system with $r = 1$, they do not degenerate and do not transform, for example, into the equations of motion of a point on the surface Σ' under the action of the force F.

2. THE EQUATIONS OF MOTION

Let us substitute the expression for the reaction R following from the first equation of system (1.2) into the second equation. We obtain

$$M\dot{w} = -N \times (m\dot{v} - F) = -N \times \left(m\frac{d}{dt}(w \times N) - F\right) = -N \times (m((\dot{w} \times N) + (w \times \dot{N})) - F)$$

where $\dot{N}$ is the vector with components $(\partial N_i/\partial x^j)v_j (i, j = 1, 2, 3)$ and x^i are Cartesian coordinates.

Since $M\dot{w} = -N \times R$, the derivative $\dot{w}$ is orthogonal to the normal N; consequently,

$$N \times (\dot{w} \times N) = \dot{w}$$

We therefore obtain from the preceding equality

$$(M + m)\dot{w} = -mN \times (w \times \dot{N}) + N \times F \tag{2.1}$$

Now define a scalar variable $u = (w, N)$, namely, the projection of the angular velocity of the ball onto the normal to the surface. Note that

$$w = uN - v \times N, \quad N \times (w \times \dot{N}) = -u\dot{N}$$

The vector product of Eqs (2.1) by N (on the right) is

$$(M + m)(\dot{w} \times N) = mu\dot{N} \times N + (N \times F) \times N$$

Hence it follows that

$$\dot{w}\times N = \frac{d}{dt}(w\times N) - w\times\dot{N} = \dot{v} - (uN - v\times N)\times\dot{N}$$

Thus,

$$(M+m)(\dot{v} - (uN - v\times N)\times\dot{N}) = mu\dot{N}\times N + (N\times F)\times N$$

Since $(N, \dot{N}) = 0$, it follows that $(v\times N)\times\dot{N} = (v, \dot{N})N$, and

$$\dot{v} + (v, \dot{N})N = \frac{M}{M+m}uN\times\dot{N} + \frac{1}{M+m}(F - (F, N)N) \tag{2.2}$$

We now multiply Eqs (2.1) scalarly by N. This gives

$$(M+m)(\dot{w}, N) = 0$$

Therefore

$$\dot{u} = \frac{d}{dt}(w, N) = (w, \dot{N})$$

Consequently

$$\dot{u} = ((\dot{N}\times N), v) \tag{2.3}$$

Together with the relation $\dot{x} = v$, Eqs (2.2) and (2.3) from a closed system of equations. If we formally put $M = 0$, Eq. (2.2) will describe the motion of a point of mass m on the surface Σ' under the action of the force F. System (2.2), (2.3) has also been obtained by different arguments, in the case when F is the force of gravity [7].

Remark 2. Let F be a potential force. Then system (2.2), (2.3) has a first integral (the energy integral). It is obtained by the standard procedure: take the scalar product of Eq. (2.2) by $(M + m)v$ and that of Eq. (2.3) by Mu, add, and use the fact that $(N, v) = 0$.

3. INVARIANT MEASURE

Suppose the surface is (locally) defined Σ in Cartesian coordinates (x^1, x^2, x^3) by an equation $x^3 = f(x^1, x^2)$. The projection of Eq. (2.1) onto the x^1, x^2 axes gives a system

$$\ddot{x}^i + \left(\Gamma^i_{jk}\dot{x}^j\dot{x}^k - \frac{1}{M+m}g^{ij}Q_j\right) = \frac{M}{M+m}uA^i_j\dot{x}^j, \quad i, j, k = 1, 2 \tag{3.1}$$

where g^{ij} is the induced metric on the surface Σ', expressed in terms of the local coordinates (x^1, x^2) on Σ', Γ^i_{jk} are the Christoffel symbols compatible with the metric g^{ij}, and Q is the generalized force defined by projecting F onto the surface Σ'. If the right-hand sides of Eqs (3.1) were to vanish, we would obtain the usual Lagrange equations (to be solved for the accelerations) for the motion of a point on the surface Σ'. It is clear that if the original force F is a potential force, Q will also be a potential force.

Lemma 1. The quantities $A^i_j\dot{x}^j$ $(i, j, = 1, 2)$ on the right of Eqs (3.1) may be expressed as

$$A^1_j\dot{x}^j = h_1/\tilde{f}, \quad A^2_j\dot{x}^j = -h_2/\tilde{f}$$

where

$$h_1 = (f_{12}v^1 + f_{22}v^2)/\tilde{f}, \quad h_2 = (f_{11}v^1 + f_{12}v^2)/\tilde{f}, \quad \tilde{f} = \sqrt{1 + f_1^2 + f_2^2}$$

$$f_i = \partial f/\partial x^i, \quad f_{ij} = \partial^2 f/\partial x^i\partial x^j, \quad v^i = \dot{x}^i, \quad i, j = 1, 2$$

The proof proceeds by direct calculation: we have to find the first two components of the three-dimensional vector $\dot{N}\times N$. We have

$$N = (f_1, f_2 - 1)/\tilde{f}$$

$$\dot{N} = -(f_1 h_2 + f_2 h_1)(f_1, f_2 - 1)/\tilde{f}^2 + (h_2, h_1, 0) =$$

$$= (-f_1 f_2 h_1 + (1 + f_2^2)h_2, -f_1 f_2 h_2 + (1 + f_1^2)h_1, f_1 h_2 + f_2 h_1)/\tilde{f}^2$$

Having the components of the vectors N and $\dot{N}$, one can readily determine their vector product.

Theorem 1. Let a system of the form (3.1) (of arbitrary dimensions n) be closed by adding an equation in u:

$$\dot{u} = U(x, \dot{x}, t) \tag{3.2}$$

where U is an arbitrary smooth function of x and $\dot{x}$, and A_i^j are functions of x and t. Let the forces Q have the form

$$Q_i = \frac{d}{dt}\frac{\partial V}{\partial \dot{x}^i} - \frac{\partial V}{\partial x^i} + g_{ij} B_k^j \dot{x}^k + Q_i^*$$

where $V = b_i(x, t)\dot{x}^i + V_0(x, t)$ is a generalized potential, the matrix $B(x, t)$ has zero trace, and the non-potential forces Q_i^* are independent of the velocities. Suppose the trace of the matrix A vanishes: $\operatorname{tr} A = A_1^1 + \ldots + A_n^n = 0$. Then system (3.1), (3.2) has a smooth invariant measure.

Roughly speaking, the conditions imposed in the theorem on the generalized forces Q are satisfied if there is no dissipation in the system. For example, they are surely satisfied if system (3.1) is a natural Lagrangian system with $M = 0$, and the quadratic part of the Lagrangian is then $g_{ij}\dot{x}^i \dot{x}^j/2$.

Proof. The classical Lagrange equations of the second kind are obtained from system (3.1) by "lowering indices." Apply a Legendre transformation with respect to the velocities $\dot{x}^i$: $p_i = g_{ij}\dot{x}^j + b_i$. We obtain "perturbed Hamiltonian equations"

$$\begin{aligned} \dot{p}_i &= -\frac{\partial H}{\partial x^i} + u\tilde{A}_i^j p_j + \tilde{B}_i^j p_j + \tilde{Q}_i, \quad \dot{x}^i = \frac{\partial H}{\partial p_i}, \quad \dot{u} = U(p, x, t) \\ \tilde{A}_i^j &= g_{ik}\frac{M}{M + m} A_l^k g^{lj}, \quad \tilde{B}_i^j = g_{ik} B_l^k g^{lj} \end{aligned} \tag{3.3}$$

where the terms $\tilde{Q}_i$ are independent of the momenta p.

The divergence of the right-hand side of system (3.3) equals the same of the product of u by the trace of the matrix $\tilde{A}$ and the trace of the matrix $\tilde{B}$. But the trace of a matrix is an invariant, and therefore

$$\operatorname{tr}\tilde{A} = \frac{M}{M + m}\operatorname{tr} A = 0, \quad \operatorname{tr}\tilde{B} = \operatorname{tr} B = 0$$

Thus, the standard volume is conserved in the phase space (p, x, u).

Corollary. In the classical problem of a ball rolling on a surface and in the limiting case when the radius of the ball tends to zero, there is a smooth invariant measure if the external force satisfies the condition of the theorem (for example, if it has a generalized potential).

Proof. By Lemma 1, in the system under consideration we have $A_1^1 + A_2^2 = 0$. For the relation with the limiting case when the radius of the ball tends to zero, $r \to 0$, see Proposition 2.

Remark 3. Suppose all the functions do not depend explicitly on time and that the forces Q are potential forces. Then, in the problem of the rolling ball, the function U may be found from the condition that $H + Mu^2/2$ is a first integral (see Remark 2). Indeed, since

$$\frac{dH}{dt} + Mu\dot{u} = 0$$

it follows from Eqs (3.3) that

$$u\tilde{A}_i^j p_j \frac{\partial H}{\partial p_i} + MuU(p, x) = 0$$

Consequently

$$MU = -\tilde{A}_i^j p_j \frac{\partial H}{\partial p_i} \tag{3.4}$$

4. DISCUSSION

The theorem just proved answers the question as to the existence of an invariant measure both in the general case of a symmetrical ball rolling without slip on a surface, under fairly general assumptions as to the nature of the applied forces, and in the limiting case when the radius of the ball tends to zero. This limiting case is of independent interest, as an example of a system with "hidden motions": the motion of a particle on a surface is defined not only by local coordinates on the tangent (or cotangent) bundle, but also by a certain additional parameter u – the "spin" of the particle. From that point of view, system (3.3), (3.4) is particularly interesting as a generalization of the problem of a rolling ball.

An interesting "inverse" problem has been considered [14]: a convex body whose inertia tensor is spherical is rolling on a fixed sphere (there is no force field); the results also imply the existence of an invariant measure in the "direct" problem of a ball rolling on a convex surface, in the case when the external force depends only on the position of the system.

One can consider another limiting problem, when the radius of the ball remains unchanged but the moment of inertia tends to zero, $M \to O$. This means that the mass of the ball collapses to its centre. System (3.3) is regular as $M \to O$ (the coefficients $\tilde{A}_i^j$ are quantities of the same order as M when $M \to O$). This approach makes it possible to utilize the rich apparatus of perturbation theory to analyse the dynamics in the case of small M. It is interesting to compare system (3.3) for small values of M with so-called "weakly non-holonomic" systems, see [15].

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